Pythagorean Triples Formulas

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Abstract

There are multiple formulas to help provide three integers which satisfy the Pythagorean Theorem, $a^2 + b^2 = c^2$. The simplest formula of $a = x^2 - y^2$, $b = 2xy$, and $c = x^2 + y^2$ is easy to use but does not provide all triples that satisfy the Pythagorean Theorem. Fortunately, there are other formulas which can be used to find all Pythagorean triples. The Height-Excess Enumeration Theorem provides a formula for calculating all Pythagorean triples, while Wade-Wade and the Fibonacci sequence provide recursive formulas for calculating Pythagorean triples. This paper will concentrate on proving the Height-Excess Enumeration Theorem and provide the formulas for the two recursive formulas.

The Height-Excess Enumeration Theorem provides a formula for calculating a Pythagorean triple based on two chosen values called h and k. The formula is derived from breaking side a of the right triangle into two parts: the height and the excess. The height of side a is calculated by subtracting b from c. Thus, $h = c - b$. The excess of side a is the length leftover from subtracting h from a. The excess of the right triangle is where the value of k comes into play. The value d is called the increment. Figure 1 gives a geometric representation of the right triangle used to create the theorem.

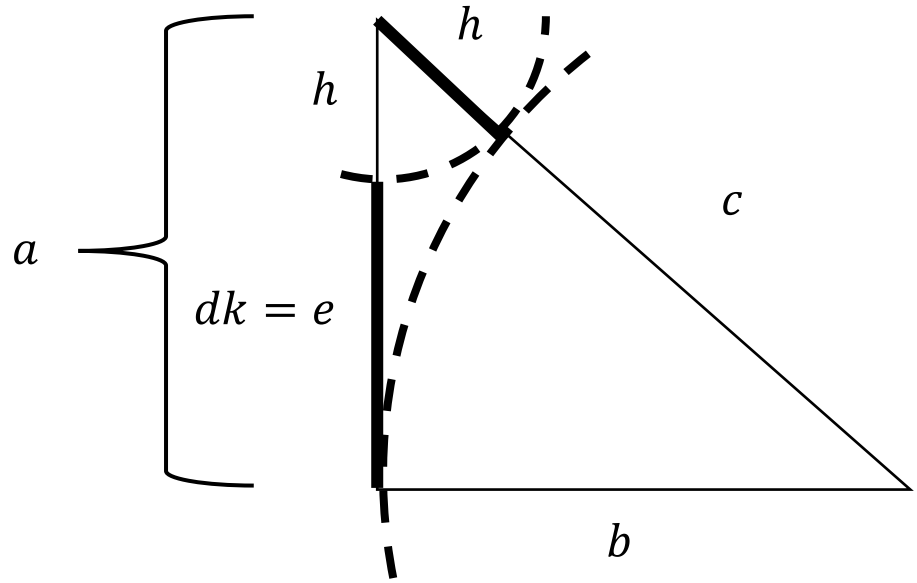

Figure 1. Right triangle broken into a height and excess.

Height-Excess Enumeration Theorem. To a positive integer h, written as pq^2 with p square-free and q positive, associate the number $d = \begin{cases} 2pq & \text{if } p \text{ is odd} \\ pq & \text{if } p \text{ is even} \end{cases}$. As one takes all pairs (h, k) of positive integers, the formula $P(h,k) = \left(h + dk,\ dk + \frac{(dk)^2}{2h},\ h + dk + \frac{(dk)^2}{2h}\right)$ produces each Pythagorean triple exactly once.

Let us start with checking the formula to see it supports the Pythagorean Theorem. Squaring each part of the formula gives $a^2 = (h + dk)^2 = h^2 + 2hdk + d^2k^2$, $b^2 = \left(dk + \frac{(dk)^2}{2h}\right)^2 = d^2k^2 + \frac{d^3k^3}{h} + \frac{d^4k^4}{4h^2}$, and $c^2 = \left(h + dk + \frac{(dk)^2}{2h}\right)^2 = h^2 + d^2k^2 + + \frac{d^4k^4}{4h^2} + 2hdk + d^2k^2 + + \frac{d^3k^3}{h}$. Each term in c^2 appears in either a^2 or b^2 with no excess or missing terms. Thus, the formula holds true for the Pythagorean Theorem.

To prove the theorem, the following lemma is required to establish properties of the increment, d.

Lemma. Let h be a positive integer with associated increment d. Then $2h|d^2$. If D is any positive integer for which $2h|D^2$, then $d|D$.

Proof:

- Prove $2h|d^2$. Case 1-Suppose p is even. By definition $d = pq$. Then $d^2 = p^2q^2 = 2pq^2\left(\frac{p}{2}\right) = 2h\left(\frac{p}{2}\right)$. Thus $2h|d^2$ because $\frac{p}{2}$ is an integer since p is even. Case 2-Suppose p is odd. By definition $d = 2pq$. Then $d^2 = 4p^2q^2 = 2pq^2(2p) = 2h(2p)$. Thus $2h|d^2$ because $2p$ is an integer since p is an integer. Therefore, $2h|d^2$.
- Prove if $2h|D^2$, then $d|D$. Suppose $2h|D^2$. By The Fundamental Theorem of Arithmetic let $D = d_1^{r_1} \cdots d_\ell^{r_\ell}$, $p = p_1 \cdots p_m$, and $q^2 = q_1^{2t_1} \cdots q_n^{2t_n}$, where $d_\ell^{r_\ell}$, p_m, and $q_n^{2t_n}$ are distinct prime factors respectively. By the definition of divide $D^2 = 2hk$, for some integer k. Then $d_1^{2r_1} \cdots d_\ell^{2r_\ell} = 2p_1 \cdots p_m \cdot q_1^{2t_1} \cdots q_n^{2t_n} \cdot k$. Each q_i must equal a d_j such that $2t_i \leq 2r_j$, then $t_i \leq r_j$. Thus, $q|D$ and $D = qk_1$ for some integer k_1. Then $D^2 = 2pq^2k$ and $D^2 = q^2k_1^2$, which gives $2pq^2k = q^2k_1^2$. Then $2pk = k_1^2$ and k_1^2 is even. Let $k_1 = s_1^{u_1} \cdots s_w^{u_w}$ where $s_w^{u_w}$ are distinct prime factors. Then $2p_1 \cdots p_m k = s_1^{2u_1} \cdots s_w^{2u_w}$ and each p_i must equal one of the s_j. Thus $p|k_1$

and $k_1 = pk_2$ for some integer k_2. Case 1- Suppose p is even, then $d = pq$. By substitution $D = qk_1 = q \cdot pk_2 = pq \cdot k_2 = dk_2$. Thus, $d|D$. Case 2-Suppose p is odd, then $d = 2pq$. Since k_1^2 is even, then k_1 is even. Thus, $2|k_1$ which gives $2|k_1^2$ and 2 is one of the s_j. Then $2p|k_1$ and $k_1 = 2pk_2$ for some integer k_2. By substitution $D = qk_1 = q \cdot 2pk_2 = 2pq \cdot k_2 = dk_2$. Thus, $d|D$.

With this lemma we can prove the Height-Excess Enumeration Theorem.

Proof: Let (a, b, c) be a Pythagorean triple, then $a^2 + b^2 = c^2$. Let $h = c - b$ and $e = a - h = a - (c - b) = a + b - c$. This gives three equations which can be used to solve for a, b, and c in terms of h and e. First $h + e = (c - b) + (a + b - c)$, which simplifies to $a = h + e$. Then $a^2 = c^2 - b^2$ gives $(h + e)^2 = (c - b)(c + b)$, which gives $h^2 + 2he + e^2 = h(c + b)$ through multiplication and substitution. Thus $\begin{cases} c + b = h + 2e + \frac{e^2}{h} \\ c - b = h \end{cases}$. By elimination method $2c = 2h + 2e + \frac{e^2}{h}$, which simplifies to $c = h + e + \frac{e^2}{2h}$. By substitution $e = (h + e) + b - \left(2h + 2e + \frac{e^2}{h}\right)$. Then solving for b gives $b = e + \frac{e^2}{2h}$. Start with substituting the values for a, b, and c into $2(c - a)(c - b)$. Then $2(c - a)(c - b) = 2\left(h + e + \frac{e^2}{2h} - (h + e)\right)\left(h + e + \frac{e^2}{2h} - \left(e + \frac{e^2}{2h}\right)\right) = 2 \cdot \frac{e^2}{2h} \cdot h = e^2$. Thus $e^2 = 2(c - a) \cdot h = 2h \cdot (c - a)$. Since a and c are integers, $c - a$ is an integer and $2h|e^2$. Then $d|e$ by the previous lemma, which gives $e = dk$ for some integer k. Therefore $(a, b, c) = P(h, k) = \left(h + dk,\ dk + \frac{(dk)^2}{2h},\ h + dk + \frac{(dk)^2}{2h}\right)$.

This theorem also has an extension which gives the conditions for a primitive Pythagorean triple and a Pythagorean triangle. A primitive Pythagorean triple is a triple that cannot be reduced to a smaller triple, thus GCD(a, b, c) is equal to 1. A Pythagorean triangle is a triple in which $a < b$.

Theorem. The primitive Pythagorean triples occur exactly when $\mathrm{GCD}(h,k) = 1$ and either $h = q^2$ with q odd, or $h = 2q^2$. The Pythagorean triangles occur exactly when $k > \frac{h}{d}\sqrt{2}$.

Proof:

- primitive Pythagorean triples.

Case 1-Let (a,b,c) be a primitive triple. Suppose r is a prime dividing $c-a$ and $c-b$. Then r divides $(c-a)^2 + (c-b)^2 = c^2 - 2ac + a^2 + c^2 - 2bc + b^2 = 2c^2 - 2ac - 2bc + c^2 = 3c^2 - 2ac - 2bc = c(3c - 2a - 2b)$. Thus either $r|c$ or $r|3c-2a-2b$. If $r|c$, then contradiction since (a,b,c) is a primitive triple. If $r|3c-2a-2b$, then $r|c$ since $3c - 2a - 2b = 2c - 2a + c - 2b + c - c = 2(c-a) + 2c - 2b + c = 2(c-a) + 2(c-b) + c$. This gives a contradiction again and $c-a$ and $c-b$ are relatively prime. By substitution $c - a = h + dk + \frac{(dk)^2}{2h} - (h+dk) = \frac{(dk)^2}{2h}$ and $c - b = h + dk + \frac{(dk)^2}{2h} - \left(dk + \frac{(dk)^2}{2h}\right) = h$.

Case a-Suppose p is odd, then $d = 2pq$. Thus $c - a = \frac{(dk)^2}{2h} = \frac{(2pqk)^2}{2pq^2} = 2pk^2$ and $c - b = h = pq^2$. Since $\mathrm{GCD}(c-a, c-b) = 1$, then $p = 1$, $h = q^2$, and $\mathrm{GCD}(2k^2, q^2) = 1$. Since $2k^2$ is even, q^2 must be odd to be relatively prime. Thus q is odd and $\mathrm{GCD}(h,k) = 1$.

Case b-Suppose p is even, then $d = pq$. Thus $c - a = \frac{(dk)^2}{2h} = \frac{(pqk)^2}{2pq^2} = \frac{p}{2}k^2$ and $c - b = h = pq^2$. Since $\mathrm{GCD}(c-a, c-b) = 1$, then $p = 2$, $h = 2q^2$, and $\mathrm{GCD}(k^2, 2q^2) = 1$. Thus $\mathrm{GCD}(h,k) = 1$.

Case 2.A-Let $\mathrm{GCD}(h.k) = 1$ and $h = q^2$ with q odd. Thus $p = 1$ and $d = 2q$. Then $a = h + dk = q^2 + 2kq = q(q+2k)$ and $b = dk + \frac{(dk)^2}{2h} = 2kq + \frac{(2kq)^2}{2q^2} = 2kq + 2k^2 = 2k(q+k)$. Let r be a prime sch that $r|a$ and $r|b$. Since q and $q+2k$ are odd, then a is odd and r must be odd. Then $r|q(q+2k)$ and $r|2k(q+k)$.

Case a-Suppose $r|q$ and $r|2k$. Since r is odd, $r|k$. Thus $r|h$ and $r|k$, which is a contradiction.

Case b- Suppose $r|q$ and $r|q+k$. Then $r|k$. Thus $r|h$ and $r|k$, which is a contradiction.

Case c-Suppose $r|q+2k$ and $r|2k$. Then $r|q$ and $r|k$ since r is odd and cannot divide 2. Thus $r|h$ and $r|k$, which is a contradiction.

Case d- Suppose $r|q+2k$ and $r|q+k$. Then $r|q$ and $r|k$ since r is odd and cannot divide 2. Thus $r|h$ and $r|k$, which is a contradiction. Therefore (a, b, c) is primitive.

Case 2.B- Let GCD$(h. k) = 1$ and $h = 2q^2$. Then $p = 2$ and $d = 2q$. Thus $a = h + dk = 2q^2 + 2kq = 2q(q+k)$ and $b = dk + \frac{(dk)^2}{2h} = 2kq + \frac{(2kq)^2}{4q^2} = 2kq + k^2 = k(2q+k)$. Let r be a prime such that $r|a$ and $r|b$. Since h is even, k is odd. Then $2q+k$ is odd which lead to b is odd. Thus, r is odd and cannot equal 2. Then similarly to the cases above $r|h$ and $r|k$, which is a contradiction. Therefore (a, b, c) is primitive

- Pythagorean triangles.

Let $a < b$. Then $h + dk < dk + \frac{(dk)^2}{2h}$

$$\Rightarrow h < \frac{(dk)^2}{2h}$$

$$\Rightarrow 2h^2 < d^2k^2$$

$$\Rightarrow k^2 > \frac{2h^2}{d^2}$$

$$\Rightarrow k > \frac{h}{d}\sqrt{2}$$

The last two formulas give a recursive method of calculating Pythagorean triples. The first formula was developed by P.W. Wade and W.R. Wade. Their recursion is based on the Height-Excess Enumeration formula and uses the same values of h and k; $(a_{k+1},\ b_{k+1},\ c_{k+1}) =$

$\left(a_k + d,\ \frac{d}{h}a_k + b_k + \frac{d^2}{2h},\ \frac{d}{h}a_k + c_k + \frac{d^2}{2h}\right)$. The second recursion formula was given by Horadam and starts with a generalized Fibonacci sequence; let r and s be positive integers and $H_1 = r, H_2 = s$, and $H_n = H_{n-1} + H_{n-2}$. Then the triples $(2H_{n+1}^2 + 2H_nH_{n+1},\ H_n^2 + 2H_nH_{n+1},$ $H_n^2 + 2H_nH_{n+1} + 2H_{n+1}^2)$ are Pythagorean triples with a even.

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